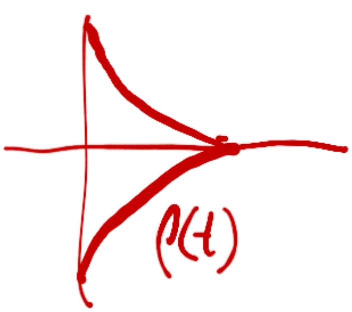
Example on line integrals for scalar functions:

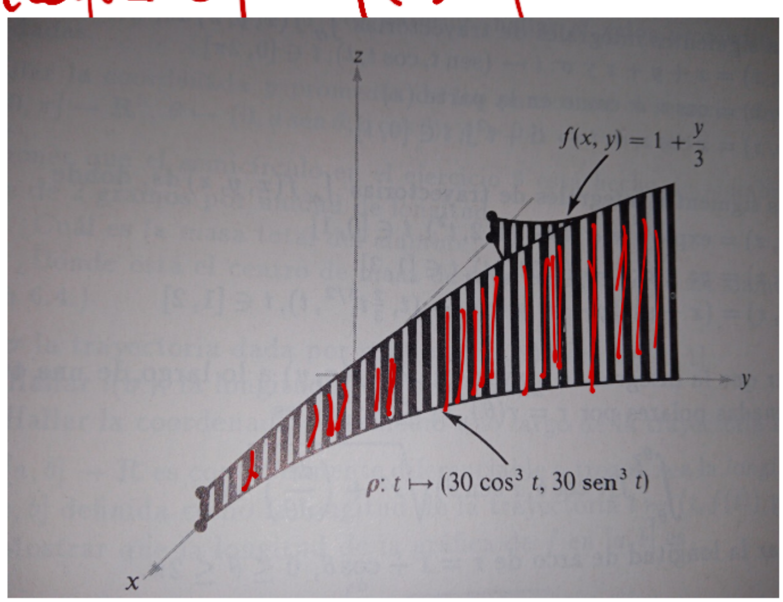
The aunt of Tom Sawyer asked him to paint both sides of her fence.

He thinks she should pay 5 cents every 25 feet^2

How much can Tom earn with this work under these conditions?

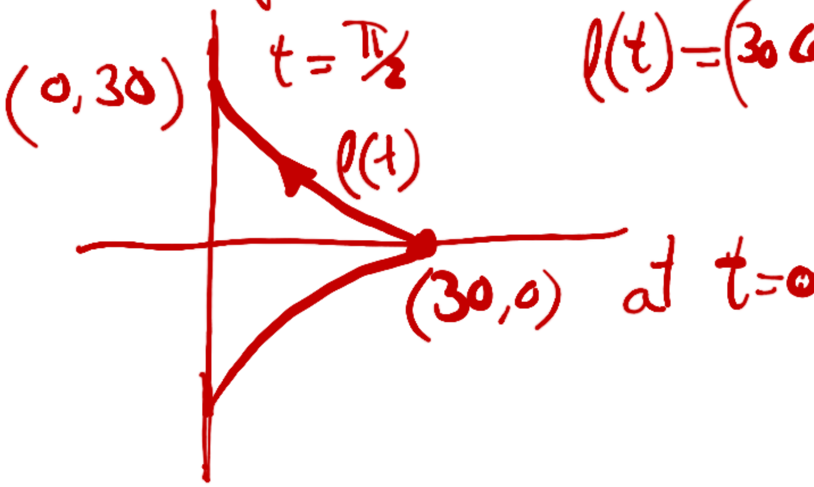
Fence is given by a function $f(x,y) = 1 + \frac{y}{3}$ along a trajectory $\rho(t) = (30\cos^3 t,\ 30\sin^3 t)$





We have $\ell(t)$, $I = [0, \frac{\pi}{2}]$ just considering the first quadrant.

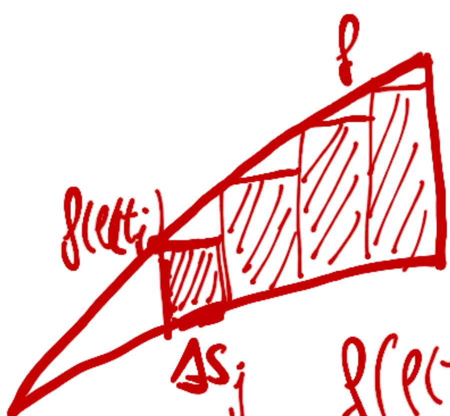
$\ell(t) = (30\cos^3 t,\ 30\sin^3 t)$



We might take a partition of $[a,b]$

$a = t_0 < \dots < t_k = b$

$\Delta t = \frac{b-a}{k}$



Area for the partition:

$f(\ell(t_j)) \cdot \Delta S_j$ = area of rectangle.

Adding all of those areas

$$A = \sum_{j=1}^{k} f(\rho(t_j)) \cdot \Delta S_j = \sum_{j=1}^{k} \underbrace{f(\rho(t_j))\, \|\rho'(t)\| \Delta t}_{\text{mean value theorem.}}$$

$$\Big\downarrow \; \Delta S_j \to 0$$

$$\int_a^b f(\rho(t))\, \|\rho'(t)\|\, dt$$

$$f(x,y) = 1 + \frac{y}{3}$$

$$\rho(t) = (30\cos^3 t, 30\sin^3 t) \qquad t \in [0, \tfrac{\pi}{2}]$$

$$f(\rho(t)) = 1 + 10\sin^3 t$$

$$\rho'(t) = 90(-\cos^2 t \cdot \sin t, \cos t \sin^2 t)$$

$$\|\rho'(t)\| = 90\sqrt{\cos^4 t \sin^2 t + \cos^2 t \sin^4 t} = 90 \cos t \sin t$$

$$\int_0^{\pi/2} (1+10\sin^3 t)\, 90 \sin t \cos t \, dt$$

$$= 90 \int_0^{\pi/2} \left(\underline{\sin t} \cos t + 10 \underline{\sin^4 t} \cos t\right) dt$$

$$= 90 \left(\frac{\sin^2 t}{2} + 2\sin^5 t\right)\Bigg]_0^{\pi/2} = 90\left(\frac{1}{2} + 2\right)$$

$$= \frac{450}{2} = 225 \quad \left(\text{one side of the fence in the first quadrant.}\right)$$

Total area $= 4 \cdot 225 = \underline{900}$ feet^2

Tom earns $= \dfrac{900}{25} \cdot 5 = \dfrac{900}{5}$

Path integrals for vector fields

$\sigma : [a,b] \longrightarrow \mathbb{R}^N$ trajectory.

$F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ vector field.

We would like to compute the line of F along $\sigma([a,b]) \equiv$ curve.

$$F : \sigma([a,b]) \longrightarrow \mathbb{R}^N$$

Definition

scalar product.

$$\int_\sigma F = \int_a^b \underbrace{F(\sigma(t))}_{\text{vector}} \cdot \underbrace{\sigma'(t)}_{\text{vector}} \, dt$$

Notation $\displaystyle\int_\sigma F = \int_\sigma F(x_1 \ldots x_N) \cdot ds = \int_\sigma \underbrace{F_1 dx_1 + \cdots + F_N dx_N}_{F \cdot ds}$

$ds = \sigma'(t)\,dt$

Example: Find the path integral of

$$F(x,y,z) = (e^y, e^x, e^z)$$

along $\sigma(t) = (0, t, t^2)$ $\quad t \in [0, \log 2]$

$$\int_\sigma F = \int_0^{\log 2} F(\sigma(t)) \cdot \sigma'(t)\, dt$$

$$F(\sigma(t)) = (e^t, 1, e^{t^2})$$

$$\sigma'(t) = (0, 1, 2t)$$

$$\int_\sigma F = \int_0^{\log 2} \left(1 + 2te^{t^2}\right) dt = \left(t + e^{t^2}\right)_0^{\log 2}$$

$$= \log 2 + e^{(\log 2)^2} - 1$$

Relation between line integrals for scalar fields and vector fields

$\sigma : [a,b] \longrightarrow \mathbb{R}^n$ trajectory.

$$\int_\sigma \mathbf{F} \cdot d\mathbf{s} = \boxed{\int_a^b \mathbf{F}(\sigma(t)) \cdot \sigma'(t)\, dt}$$

scalar product

$$= \int_a^b \underbrace{\left(\mathbf{F}(\sigma(t)) \cdot \frac{\sigma'(t)}{\|\sigma'(t)\|} \right)}_{g(\sigma(t)) \text{ scalar function.}} \cdot \|\sigma'(t)\|\, dt$$

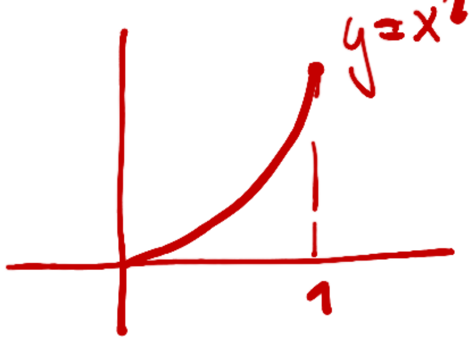
multiplication of numbers

$$= \boxed{\int_a^b g(\sigma(t)) \cdot \|\sigma'(t)\|\, dt}$$

Parametrizations

We do not have a unique parametrization.

Example: $\{(t,t^2),\ t\in[0,1]\}$



If $\left.\begin{array}{l} x=t \\ y=t^2 \end{array}\right\}\ t\in[0,1]$

$$\{(2t,(2t)^2),\ t\in[0,\tfrac{1}{2}]\}$$

Both represent the same curve.

What happens with the line integral depending on the parametrization?

Proposition

Two parametrizations of the same curve σ, ρ

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$ scalar field.

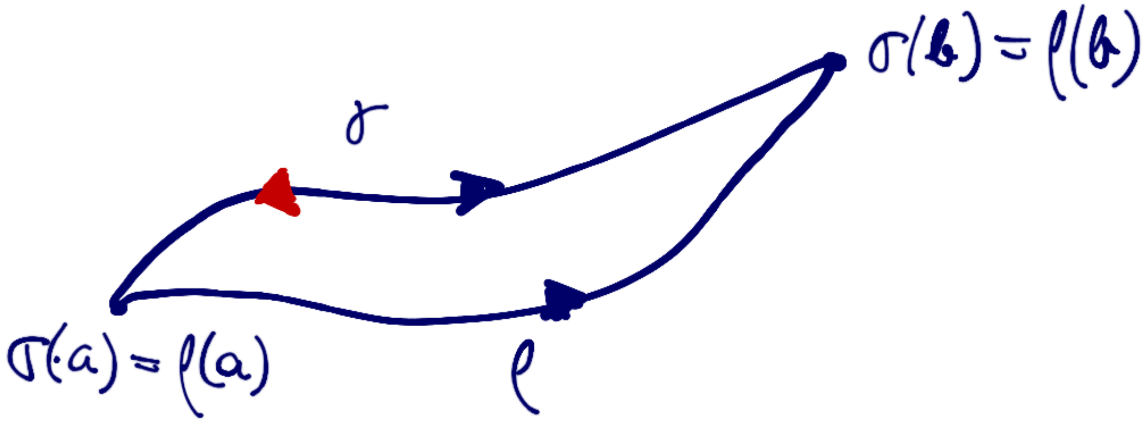
Then,
$$\int_\sigma f = \int_\rho f$$

Proposition

Two paramt. σ and ρ. $F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ vector field.

a) If σ and ρ have the same orientation
$$\int_\sigma \mathbf{F} = \int_\rho \mathbf{F}$$

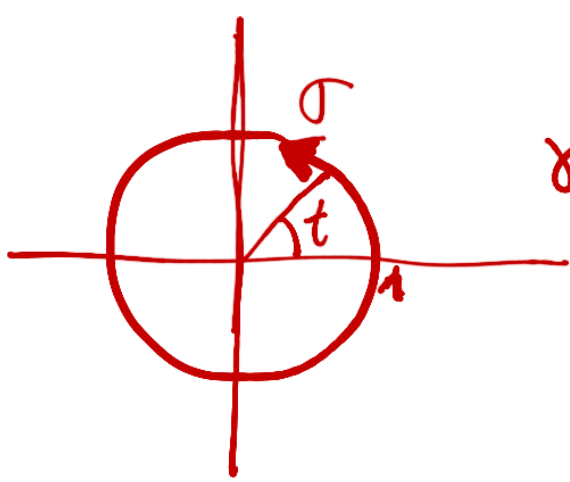
b) If σ and ρ have opposite direction
$$\int_\sigma \mathbf{F} = -\int_\rho \mathbf{F}$$



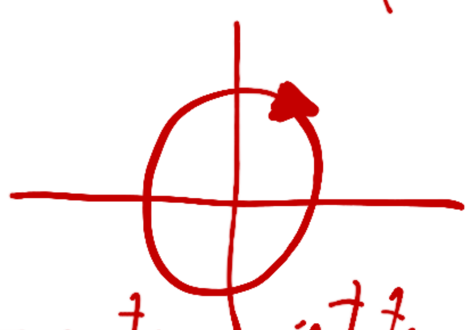
Example: Circle $\{x^2+y^2=1\}$

$$\sigma : [0, 2\pi] \to \mathbb{R}^2$$
$$t \to (\cos t, \sin t) = \sigma(t)$$

anticlockwise $\equiv$ positive orientation



$$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$$
$$t \longrightarrow \gamma(t) = (\cos(2\pi - t), \sin(2\pi - t))$$



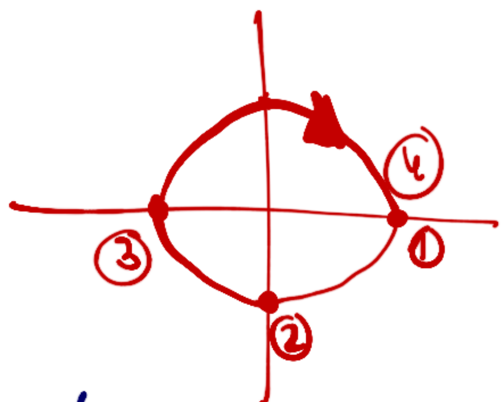
clockwise $\equiv$ negative orientation.

at $t=0$ $\gamma(0)=(1,0)$

at $t=\frac{\pi}{2}$ $\gamma\left(\frac{\pi}{2}\right)=(0,-1)$

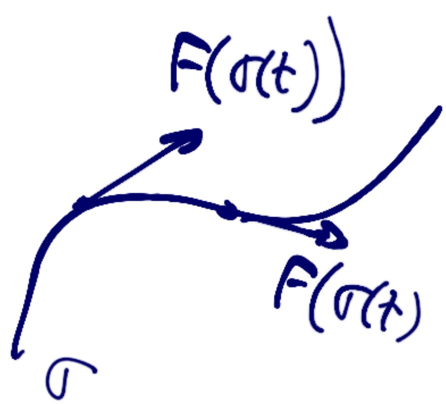
at $t=\pi$ $\gamma(\pi)=(-1,0)$

at $t=2\pi$ $\gamma(2\pi)=(1,0)$



$$\int_\sigma f = \int_a^b f(\sigma(t))\cdot \underline{\|\sigma'(t)\|}\, dt$$

$$\int_\sigma F = \int_a^b F(\sigma(t))\cdot \underline{\sigma'(t)}\, dt$$



Fundamental Theorem of Calculus

$\sigma : [a,b] \longrightarrow \mathbb{R}^N$ trajectory, $\sigma \in C^1$

and

$f : \sigma([a,b]) \longrightarrow \mathbb{R}^N$ scalar field along σ

$f \in C^1$

Then,

$$\int_\sigma \nabla f \cdot ds = \underline{f(\sigma(b)) - f(\sigma(a))}$$

Remark

- Important because $\nabla f \equiv$ vector field.

So we can define

$$\boxed{F = \nabla f}$$

If σ is a closed curve $\int_\sigma \nabla f \cdot ds = \int_\sigma F \cdot ds = 0$

Example: We would like to compute

$$\int_\sigma \mathbf{F}\cdot d\mathbf{s} = \int_\sigma \overbrace{y\,dx + x\,dy}^{F\cdot ds} \quad \text{with } \sigma(t) = \left(t^9, \sin^9\left(\frac{\pi t}{2}\right)\right)$$

$$t \in [0,1]$$

$$F(x,y) = (y, x)$$

$$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad \text{vector field.}$$

$$\int_\sigma F\cdot ds = \int_0^1 F(\sigma(t))\cdot \sigma'(t)\,dt = \int_0^1 \left(9t^8 \underline{\sin^9\left(\frac{\pi t}{2}\right)} + \dots\right)$$

$$\begin{cases} F(\sigma(t)) = \left(\underline{\sin^9\left(\frac{\pi t}{2}\right)}, t^9\right) \\ \sigma'(t) = \left(9t^8, \underline{9\sin^8\left(\frac{\pi t}{2}\right)}\cos\left(\frac{\pi t}{2}\right)\cdot\frac{\pi}{2}\right) \end{cases}$$

Substituting we get to a very complicated integral.

Looking the vector field.

$$\boxed{F(x,y) = \nabla f(x,y)} \quad \text{with} \quad f(x,y) = xy$$

$$F(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y, x)$$

Applying FTC. FTC.

$$\underbrace{\int_\sigma F \cdot ds}_{\boxed{1}} = \boxed{\int_\sigma \nabla f \cdot ds = f(\sigma(1)) - f(\sigma(0))}$$

$$\int_0^1 \nabla f(\sigma(t)) \cdot \sigma'(t)\, dt$$

$$\sigma(1) = (1,1) \Rightarrow f(\sigma(1)) = f(1,1) = 1$$

$$\sigma(0) = (0,0) \Rightarrow f(\sigma(0)) = f(0,0) = 0$$

In 1D

$$\int f(x)\,dx = F(x) + C$$ antiderivative.

$$\underbrace{F'(x) = f(x)}$$

$$\Downarrow$$

$$\int F'(x)\,dx$$

Barrow's law

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

equivalent to

$$\int_\sigma \nabla f \cdot ds = f(\sigma(b)) - f(\sigma(a))$$

Definition

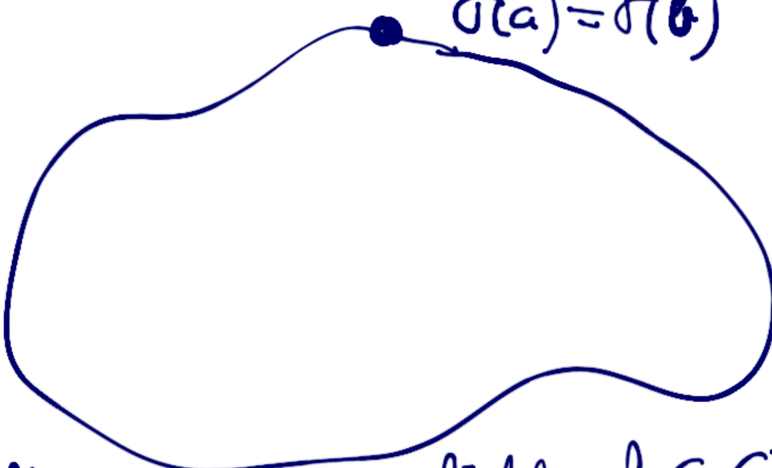
If $F = \nabla f$ then we say that F is a conservative field

Corollary (consequence)

$\sigma : [a,b] \longrightarrow \mathbb{R}^N$ trajectory of a closed curve

$\sigma(a) = \sigma(b)$ $\quad \sigma([a,b])$

$\sigma \in C^1$



$f : \mathbb{R}^N \longrightarrow \mathbb{R}$ scalar field, $f \in C^1$

Then $\boxed{\int_\sigma \nabla f = 0} = f(\sigma(b)) - f(\sigma(a))$

FTC.

Definition

$F : \Omega \longrightarrow \mathbb{R}^N$ vector field and $\Omega \subset \mathbb{R}^N$

with F having a $\underbrace{\text{exact differential form}}_{\text{conservative field.}}$

Then, there exists a scalar field

$$f : \mathbb{R}^N \longrightarrow \mathbb{R}.$$

such that $\nabla f = F$ in Ω

f is called a potential function.

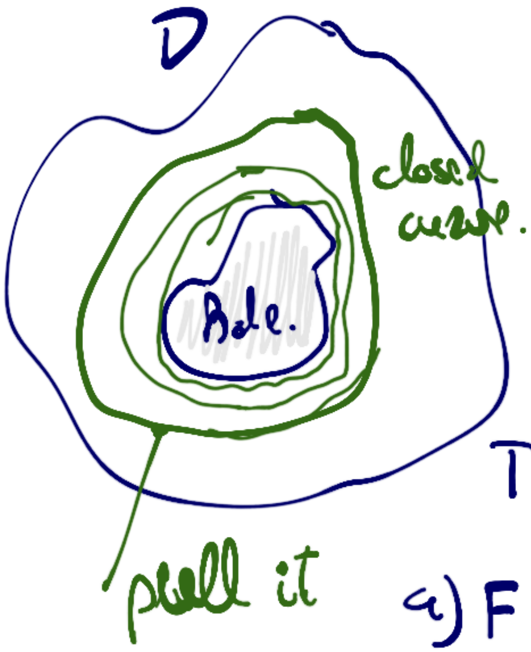
Ex: F force.

$$\underbrace{F = -\nabla V}_{\text{potential force.}}, \quad V = \text{potential function.}$$

Theorem

D is an open set simply connected

- any closed curve can be deformed continuously inside D
- In $\mathbb{R}^2, \mathbb{R}^3$ this means there are no holes.



$F \in C^1(D)$ vector field.

Then the following is equivalent:

a) F is conservative $\nabla f = F$

b) For any closed curve $\int_\sigma F = 0$

c) If we have two different parametrizations
$$\int_{\sigma_1} F = \int_{\sigma_2} F \quad F \text{ is conservative.}$$

d) In $\mathbb{R}^2$ if $F = (P, Q)$

then $\boxed{\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \sim \dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}}$

since $F = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right) = (F_1, F_2)$

then we have.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{Schwarz Th.}$$

e) In $\mathbb{R}^3$

$\text{curl } F = \text{rot } F = 0$ equivalent to d) in $\mathbb{R}^2$

curl F

$$\operatorname{rot} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

If we were in $\mathbb{R}^2$, $F_3 = 0$ for

$$F = (F_1, F_2, F_3) = (F_1, F_2, 0) \text{ and}$$

$F_1 = F_1(x,y)$
$F_2 = F_2(x,y)$

then

$$\operatorname{rot} F = \Big(0, \ 0, \ \underbrace{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}_{}\Big)$$

d) $\Rightarrow \operatorname{rot} F = 0$

Problem 1 - set 4.2

$$F(x,y,z) = \left(\sin y + z,\ x\cos y + e^z,\ x + ye^z\right)$$

a) Show that F is conservative.

$$\operatorname{rot} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x\cos y + e^z & x + ye^z \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}\left(x + ye^z\right) - \frac{\partial}{\partial z}\left(x\cos y + e^z\right)\right)$$

$$-j\left(\frac{\partial}{\partial x}\left(x + ye^z\right) - \frac{\partial}{\partial z}\left(\sin y + z\right)\right)$$

$$+k\left(\frac{\partial}{\partial x}\left(x\cos y + e^z\right) - \frac{\partial}{\partial y}\left(\sin y + z\right)\right) =$$

$$= i\left(e^z - e^z\right) - j\left(1-1\right) + k\left(\cos y - \cos y\right)$$

$$= 0 \quad (\text{vector})$$

$$\Downarrow$$

F is conservative $\Longrightarrow$ $\exists\, \phi$ such that

$$F = \nabla \phi$$

b) Compute $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$.

$$F = \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

$$= \left(\sin y + z, \ x\cos y + e^z, \ x + y e^z \right)$$

so that

$$\boxed{\frac{\partial \phi}{\partial x} = \sin y + z, \quad \frac{\partial \phi}{\partial y} = x\cos y + e^z, \quad \frac{\partial \phi}{\partial z} = x + y e^z}$$

$$\frac{\partial \phi}{\partial x} = \sin y + z \Rightarrow \phi(x,y,z) = \int (\sin y + z)\,dx$$

$$\phi(x,y,z) = \boxed{x\sin y} + \boxed{xz} + A(y,z)$$

$$\frac{\partial \phi}{\partial y} = x\cos y + e^{z} \Rightarrow \phi(x,y,z) = \int (x\cos y + e^{z})\,dy$$

$$\phi(x,y,z) = \boxed{x\sin y} + \boxed{ye^{z}} + B(x,z)$$

$$\frac{\partial \phi}{\partial z} = x + ye^{z} \Rightarrow \phi(x,y,z) = \int (x + ye^{z})\,dz$$

$$\phi(x,y,z) = \boxed{xz} + \boxed{ye^{z}} + \boxed{C(x,y)}$$

$\phi(x,y,z) = x\sin y + xz + ye^{z} + K$ Potential function